Another useful reparameterisation to obtain samples from conditional inverse Wishart distributions

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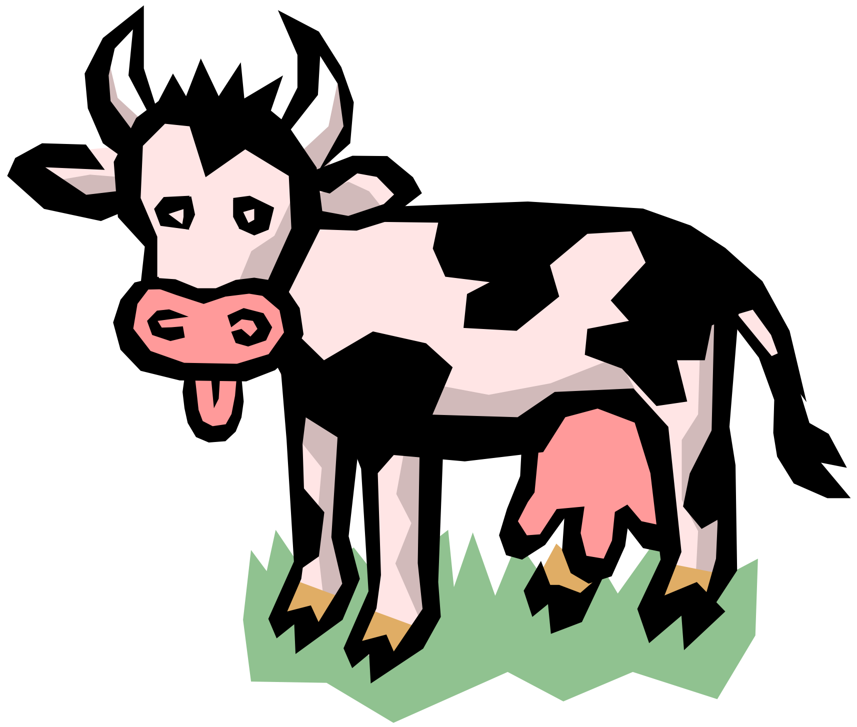
Milk yield
Mastitis
Ketosis



Animals with record

One Gaussian trait and two binary threshold traits

Milk yield
Mastitis 0/1
Ketosis 0/1

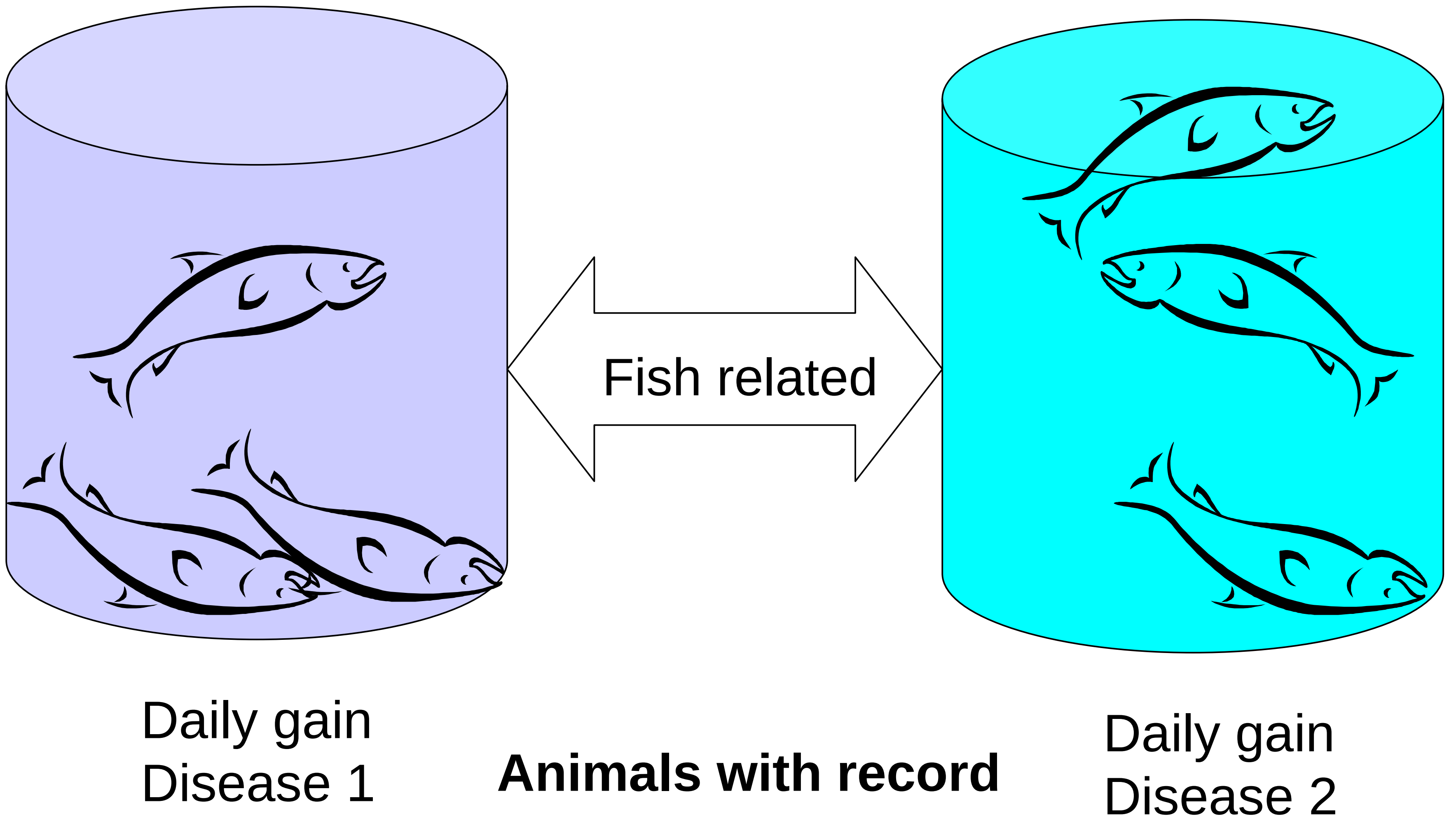
$$e_i \sim N_3(0, r_c)$$

$$r_c = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & 1 & r_{23} \\ r_{31} & r_{32} & 1 \end{bmatrix}$$

Bayesian analysis using Gibbs sampling

$$e_i|(R = r_c) \sim N_3(0, r_c)$$

- **A priori R conditional Inverse Wishart (IW)**
- **Fully conditional posterior distribution of R is conditional IW**
- **The conditioning is on** $R_{22} = R_{33} = 1$
- **Difficult to sample from**



Fish related
Daily gain
Disease 1
Animals with record
Daily gain
Disease 2

$$r_c = \begin{array}{|c|c|c|} \hline r_{11} & r_{12} & r_{13} \\ \hline r_{21} & 1 & 0 \\ \hline r_{31} & 0 & 1 \\ \hline \end{array}$$

Bayesian analysis using Gibbs sampling

$$e_i|(R = r_c) \sim N_3(0, r_c)$$

- A priori **R** conditional Inverse Wishart (IW)
- Fully conditional posterior distribution of **R** is conditional IW
- The conditioning is on $\begin{pmatrix} R_{22} & R_{23} \\ R_{32} & R_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
- Easy to sample from

Fortunately

If we

- Reparameterise the model
- Choose another prior
- Method of decomposition

Then

- The fully conditional posterior of R is

  Easy to sample from

Reparameterisation of the model

- Instead of

$$r_{22}=r_{33}=1$$

- Take

$$e_i \sim N_3(0, r_c)$$

$$r_{22}=1, \quad r_{33}- r_{32}\, r_{22}^{-1} r_{23}=1$$

$r_c =$

r_{11}	r_{12}	r_{13}
r_{21}	1	r_{23}
r_{31}	r_{32}	$1+r^2_{23}$

Another prior

$$e_i | (R = r_c) \sim N_3(0, r_c)$$

- A priori R is conditional IW
- Fully conditional posterior of R is conditional IW
- The conditioning is on

  $R_{22}=1, \quad R_{33} - R_{32} R_{22}^{-1} R_{23} = 1$

Method of decomposition

A sampled value (x,y) from (X,Y)|Z=z
can be obtained by

1) Sampling y from Y|Z=z
2) Sampling x from X|Y=y,Z=z

Method of decomposition

A sampled value r_c from

$$R \mid (R_{22} = R_{33 \bullet 2} = 1)$$

where $R_{33 \bullet 2} = R_{33} - R_{32} R_{22}^{-1} R_{23}$

can be obtained by

Method of decomposition cont.

1) Sample $\begin{pmatrix} r_{22} & r_{23} \\ r_{32} & r_{33} \end{pmatrix}$ from $\begin{pmatrix} R_{22} & R_{23} \\ R_{32} & R_{33} \end{pmatrix} | (R_{22} = R_{33\bullet 2} = 1)$

$$\begin{pmatrix} R_{22} & R_{23} \\ R_{32} & R_{33} \end{pmatrix} = \begin{pmatrix} S_1^{-1} & -S_1^{-1}S_2' \\ -S_2S_1^{-1} & S_3^{-1} + S_2S_1^{-1}S_2' \end{pmatrix}$$

$S_1 \sim W_1, \qquad S_3 \sim W_1, \qquad S_2 | (S_3 = s_3) \sim N_1$

S_1 is independent of (S_2, S_3)

$(R_{22} = R_{33\bullet 2} = 1) \Leftrightarrow (S_1 = S_3 = 1)$

Method of decomposition cont.

2) Sample $\mathbf{r}_c$ from $\mathbf{R} \Big| \left(\begin{pmatrix} R_{22} & R_{23} \\ R_{32} & R_{33} \end{pmatrix} = \begin{pmatrix} r_{22} & r_{23} \\ r_{32} & r_{33} \end{pmatrix} \right)$

$$\mathbf{R} = \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{pmatrix} = \begin{pmatrix} T_1^{-1} + T_2 T_3^{-1} T_2' & -T_2 T_3^{-1} \\ -T_3^{-1} T_2' & T_3^{-1} \end{pmatrix}$$

$$T_1 \sim W_1, \qquad T_3 \sim W_2, \qquad T_2 | (T_1 = t_1) \sim N_2$$

T_3 is independent of (T_1, T_2)

$$\left(\begin{pmatrix} R_{22} & R_{23} \\ R_{32} & R_{33} \end{pmatrix} = \begin{pmatrix} r_{22} & r_{23} \\ r_{32} & r_{33} \end{pmatrix} \right) \Leftrightarrow \left(T_3^{-1} = \begin{pmatrix} r_{22} & r_{23} \\ r_{32} & r_{33} \end{pmatrix} \right)$$

Recipe

- Sample s_2 from $S_2|(S_3 = 1) \sim N_1$
- Then

$$\begin{pmatrix} r_{22} & r_{23} \\ r_{32} & r_{33} \end{pmatrix} = \begin{pmatrix} 1 & -s_2 \\ -s_2 & 1+s_2^2 \end{pmatrix}$$

is a sampled value from

$$\begin{pmatrix} R_{22} & R_{23} \\ R_{32} & R_{33} \end{pmatrix} |(R_{22} = R_{33\bullet 2} = 1)$$

Recipe continued

- Sample t_1 from $T_1 \sim W_1$
- Sample t_2 from $T_2|(T_1 = t_1) \sim N_2$
- Let $t_3^{-1} = \begin{pmatrix} 1 & -s_2 \\ -s_2 & 1 + s_2^2 \end{pmatrix}$

Recipe continued

- Then

$$r_c = \begin{pmatrix} t_1^{-1} + t_2 t_3^{-1} t_2' & -t_2 t_3^{-1} \\ -t_3^{-1} t_2' & t_3^{-1} \end{pmatrix}$$

is a sampled value from

$$R \,|\, (R_{22} = R_{33\bullet 2} = 1)$$

However

The models are not equivalent from a Bayesian point of view

in the sense that

the posterior distribution of parameters in parameterisation 2, Ψ, cannot be obtained from the posterior distribution of parameters in parameterisation 1, Θ, via the Transformation Theorem

$$p_{\Psi|Y}(\psi|y) \neq p_{\Theta|Y}(\theta|y)\left|\frac{\partial\theta(\psi)}{\partial\psi}\right|$$

Theorem

Any two models that are equivalent in the classical (non-Bayesian) sense, are equivalent in the Bayesian sense (a posteriori), i.e.

$$p_{\Psi|Y}(\psi|y) = p_{\Theta|Y}(\theta|y)\left|\frac{\partial\theta(\psi)}{\partial\psi}\right|$$

if and only if the corresponding priors are equivalent, i.e.

$$p_{\Psi}(\psi) = p_{\Theta}(\theta)\left|\frac{\partial\theta(\psi)}{\partial\psi}\right|$$

Conclusion

In multivariate models of

Gaussian and binary traits with residual correlated liabilities

we can easily obtain samples from

the fully conditonal posterior of R